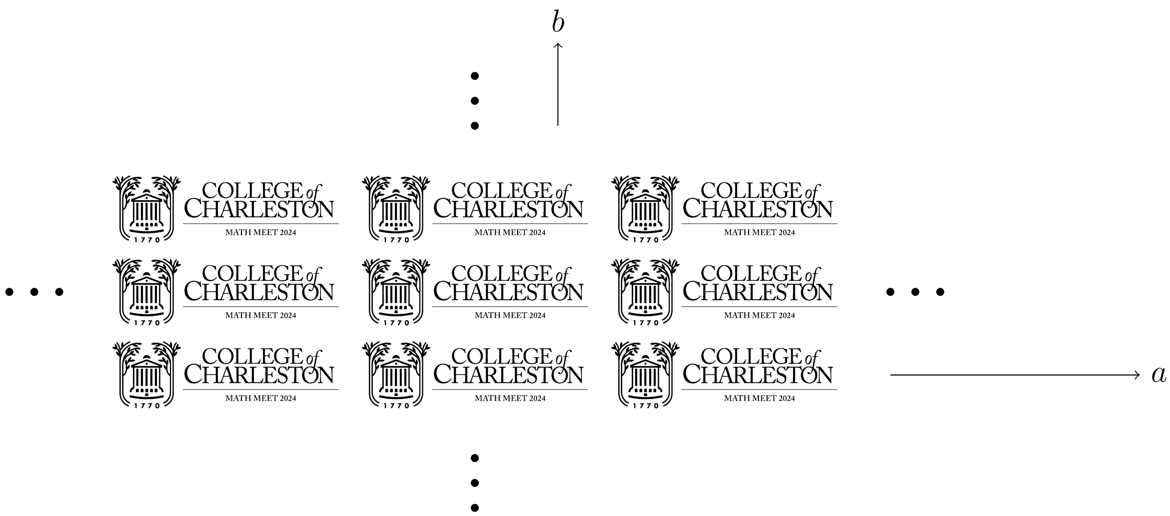
COLLEGE of CHARLESTON

MATH MEET 2024

Compile Tile Profiles

Tilings are obviously geometric things, but many aspects of a tiling can also be represented algebraically. Let's look at the symmetries of a tiling. For example, the tiling shown below has two dimensions of translational symmetries, but no rotational symmetries.



We can encode this information algebraically. Let a be the rigid motion which translates the entire plane one tile to the right. Let b be the rigid motion which translates the entire plane one tile up. Let e be the rigid motion which does nothing at all. Since these are functions, we can compose them. Then $a \circ b$ would be moving one tile up and then one tile right - remember function composition is read right to left. $a \circ a$ would be moving one tile right and then one tile right. People will usually drop the $\circ$ symbol, instead writing these as ab and aa. To compress the notation even more, people will usually write a sequence of the same operation using exponent notation, for example $aaa = a^3$. And the function that undoes a would be written as a^{-1}.

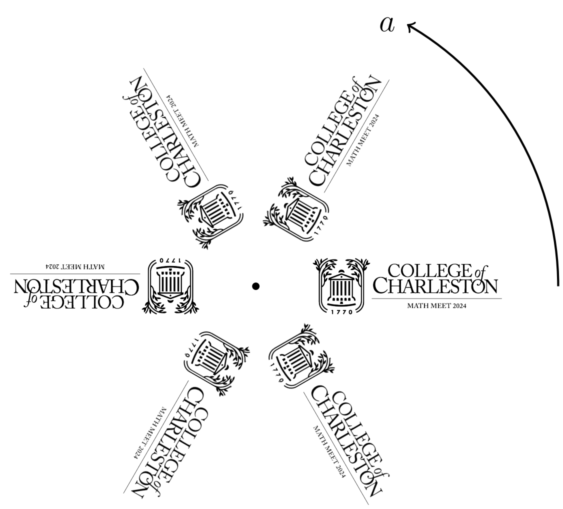
We can quickly realize some simple identities about these symmetries:

- For any function c, $ce = ec = c$
- $aa^{-1} = a^{-1}a = e$
- $bb^{-1} = b^{-1}b = e$
- $ab = ba$ (and as a consequence $ab^{-1} = b^{-1}a$, $a^{-1}b = ba^{-1}$ and $a^{-1}b^{-1} = b^{-1}a^{-1}$)

These rules actually fully describe the symmetries of the tiling. And we see that every symmetry can be written in the form $a^m b^n$ for some integers m and n (possibly 0). For example, if we start with the symmetry $baaa^{-1}bbab^{-1}a$, by repeatedly swapping the orders of a's and b's (the fourth identity), we get $aaa^{-1}aabbbb^{-1} = aaeabbe = aaabb = a^3b^2$. In particular, there are infinitely many symmetries.

In general, there's always a symmetry e that does nothing. For any other function c, $ce = ec = c$. And function composition is always associative, meaning $(fg)h = f(gh)$, so either can be written as just fgh. Any symmetry f can be undone, so has an inverse f^{-1} with the property that $ff^{-1} = f^{-1}f = e$.

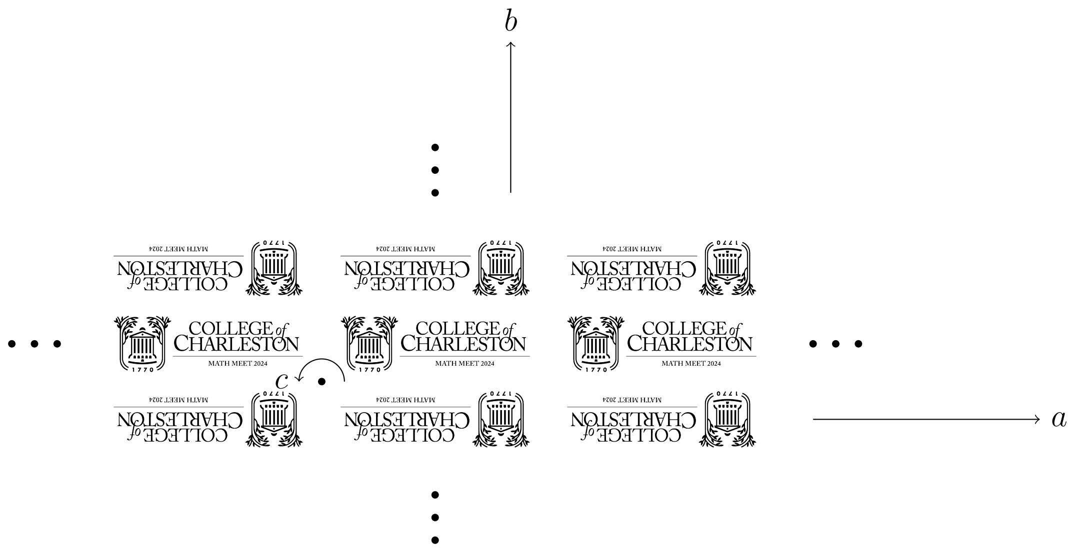
To look at another example, the tiling to the right clearly has different kinds of symmetries than the above example. If we think of a as being rotation counterclockwise by 60° around the central point, then we see that $a^6 = e$ (rotating by 360° puts every point back where it started, so is the same as doing nothing at all). In particular $a^{-1} = a^5$ (rotating 60° clockwise has the same effect as rotating by 300° counterclockwise). Every symmetry is just rotation by some multiple of 60°, so every symmetry is just some power of a. But since $a^6 = e$, we see that there are only six symmetries total: e, a, a^2, a^3, a^4, and a^5.



Let's look at some more complicated examples of symmetries.

1. The tiling below is like the first example, but also has a rotational symmetry. It can be described by the rules

   - $ab = ba$
   - $ca = a^{-1}c$, $cb = b^{-1}c$
   - $c^2 = e$



Every symmetry can be written in the standard form $a^m b^n$ or $a^m b^n c$, where m and n are integers.

(a) Using $ca = a^{-1}c$, write ca^{-1} in the standard form.

(b) Write the symmetry $a^4cb^2caca^{-1}$ in the standard form.

(c) Are the symmetries $ca^{-1}b^3ca^{-1}$ and b^{-3} the same?

2. We could also look at tilings on the sphere. The tiling shown below has symmetries that can be described by the rules
    - $a^7 = b^2 = e$
    - $ba = a^6 b$

    Again, there's a standard simplified way to write the symmetries, but I'll let you figure out what it is. Also, I haven't told you what the symmetries a and b do, just how they relate to each other.

    (a) Are the symmetries a^3ba and ba^2 the same?

    (b) There are only finitely many symmetries this time. How many are there?

3. Here's one last example, again on the sphere. The tiling shown below has symmetries that can be described by the rules
    - $a^3 = b^2 = e$
    - $aba = ba^2b$

Front view

Back view

    (a) Write bab in the form $a^m ba^n$.

    (b) Are the symmetries a^2bab and ba^2ba the same?

    (c) There are only finitely many symmetries this time. How many are there?

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Answer Sheet

1. (a) ac
   (b) $a^6b^{-2}c$
   (c) Yes

2. (a) No
   (b) 14

3. (a) a^2ba^2
   (b) Yes
   (c) 12